The Hodge decomposition theorem

Theorem Let X be a compact connected Kähler manifold. Then, for all integer $k \geq 0$, there is an isomorphism of $\mathbb{C}$-vector spaces

$$\underbrace{H^k(X;\mathbb{C})}_{\text{topological side}} \simeq \underbrace{\bigoplus_{\substack{p+q=k \\ p\geq 0,\ q\geq 0}} H^q(X;\Omega_X^p)}_{\text{complex analytic side}} .$$

$H^k(X;\mathbb{C})$: cohomology with coefficients in the locally constant sheaf $\underline{\mathbb{C}_X}$
[also the singular cohomology of X with coefficients in $\mathbb{C}$]

Ω_X^p : sheaf of holomorphic p-differential forms on X [locally free $\mathcal{O}_X$-module]

1. The Dolbeault resolution

Let $(X, \mathcal{O}_X)$ be an n-dimensional complex analytic manifold.

i.e. $(X, \mathcal{O}_X)$ is a locally ringed space which is locally isomorphic to the zero set $F^{-1}(0)$ of a holomorphic submersion

$\forall x \in X$, $\exists\ U \underset{\text{open}}{\subset} X$, $x \in U$ and $(U, \mathcal{O}_X|_U)$ isomorphic to $(\Omega, \mathcal{O}_\Omega)$

open subset of $\mathbb{C}^n$

sheaf of holomorphic functions on Ω

In holomorphic local coordinates $(z_1, \dots, z_n)$, a holomorphic p-differential form is defined as

$$\omega_z = \sum_{1 \le i_1 < \dots < i_p \le n} f_{i_1 \dots i_p}(z)\, dz_{i_1} \wedge \dots \wedge dz_{i_p}$$

with $z \longmapsto f_{i_1 \dots i_p}(z)$ holomorphic.

Equivalently, ω is a (holomorphic) section of $\Lambda^p TX$ where TX is the (holomorphic) tangent bundle of X.

This defines an $\mathcal{O}_X$-module Ω_X^p:

$$\Omega_X^p(U) := \Gamma(U; \Lambda^p TX)$$

with $(f.\omega)_z = f(z)\,\omega_z$ if $f \in \mathcal{O}_X(U)$ and $z \in U$.

Proposition The sheaf Ω_X^p is a locally free $\mathcal{O}_X$-module of rank $\binom{n}{p}$.

Proof $\Omega_X^p|_U$ has basis $(dz_{i_1} \wedge \cdots dz_{i_p})_{1 \leq i_1 < \cdots < i_p \leq n}$ as an $\mathcal{O}_X|_U$-module. $\square$

Observation
X can be seen as a $(2n)$-dimensional $\mathcal{C}^\infty$ manifold over $\mathbb{R}$, with local coordinates $x_j = \mathrm{Re}\, z_j$ and $y_j = \mathrm{Im}\, z_j$. This enables to define operators ∂ and $\bar\partial$ such that: $\forall f \in \mathcal{C}_X^\infty(U; \mathbb{C})$,

$$df = \partial f + \bar\partial f$$

with $\partial f : TU \to \mathbb{C}$ $\mathbb{C}$-linear and $\bar\partial f : TU \to \mathbb{C}$ $\mathbb{C}$-antilinear

$$\bar\partial f := \sum_{j=1}^n \frac{\partial f}{\partial \bar z_j}(z_1, \dots, z_n)\, d\bar z_j$$

with $\frac{\partial f}{\partial \bar z_j} := \frac{\partial f}{\partial x_j} + \sqrt{-1}\,\frac{\partial f}{\partial y_j}$.

For all $j \in \{1, \dots, n\}$, we have

$$z_j = x_j + \sqrt{-1}\, y_j$$

and we set

$$\overline{z_j} := x_j - \sqrt{-1}\, y_j .$$

Then $dz_j = dx_j + \sqrt{-1}\, dy_j$

and $d\overline{z_j} = dx_j - \sqrt{-1}\, dy_j$.

So $\bar\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j}$, with $\frac{\partial}{\partial \overline{z_j}} = \frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j}$.

and similarly for ∂f.

In particular, $f \in \mathcal{E}_X^\infty(U)$ is holomorphic on U iff $\bar\partial f = 0$.

Exercise

∂ and $\bar\partial$ are differential operators:

$$\partial(fg) = (\partial f) g + f(\partial g)$$

and

$$\bar\partial(fg) = (\bar\partial f) g + f(\bar\partial g).$$

The operators ∂ and $\bar{\partial}$ extend to $\mathbb{C}$-valued $\mathcal{C}^\infty$ differential forms. In particular, for all $p \geq 0$, the operator

$$\bar{\partial} : \mathcal{C}^\infty_X \longrightarrow \mathcal{A}^{0,1}_X$$

extends to a family of operators

$$\bar{\partial} : \mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p,q+1}_X$$

such that $\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\bar{\partial}\beta)$.

$\mathcal{A}^{p,q}_X$ = sheaf of $\mathbb{C}$-valued $\mathcal{C}^\infty$ differential forms of type (p,q)

If $\omega = \sum_{|I|=p,\ |J|=q} f_{I,J}(z)\, d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q} \wedge dz_{i_1} \wedge \dots \wedge dz_{i_p}$

$\mathcal{C}^\infty$ function in the variables $(z_1, \dots, z_n) \leftrightarrow (x_1, y_1, \dots, x_n, y_n)$

then $$\bar{\partial}\omega := \sum_{\substack{|I|=p \\ |J|=q+1}} \bar{\partial} f \wedge d\bar{z}_J \wedge dz_I .$$

In particular, $\omega \in \mathcal{A}^{p,0}_X(U)$ is holomorphic on U if and only $\bar{\partial}\omega = 0$.

Theorem (Dolbeault)

For all $p \geq 0$, the sequence

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_X \xrightarrow{\bar\partial} \cdots$$

is a resolution of the $\mathcal{O}_X$-module Ω^p_X, called the **Dolbeault resolution**. Moreover, it is an acyclic resolution, so

$\forall q \geq 0$,

$$H^q(X; \Omega^p_X) = \frac{\mathrm{Ker}\left(\mathcal{A}^{p,q}_X(X) \xrightarrow{\bar\partial} \mathcal{A}^{p,q+1}_X(X)\right)}{\mathrm{Im}\left(\mathcal{A}^{p,q-1}_X(X) \xrightarrow{\bar\partial} \mathcal{A}^{p,q}_X(X)\right)}$$

$$= \frac{\{\bar\partial\text{-closed } (p,q)\text{-forms}\}}{\{\bar\partial\text{-exact } (p,q)\text{-forms}\}}.$$

Sketch of a proof

- $\bar\partial$ is a morphism of $\mathcal{O}_X$-modules:

$$\bar\partial(f\omega) = (\bar\partial f)\wedge\omega + f\,\bar\partial\omega$$

so f holomorphic implies $\bar\partial(f\omega) = f\,\bar\partial\omega$.

⚠ not a morphism of $\mathcal{C}^\infty_X$-modules!

- exactness: a $\bar{\partial}$-closed form is locally exact (Dolbeault's lemma)

- acyclicity of the $\mathcal{A}_X^{p,q}$:
  $\mathcal{A}_X^{p,q}$ is a $\mathcal{C}_X^\infty$-module and $\mathcal{C}_X^\infty$ is soft.

$\square$

When X is compact, Dolbeault's theorem sets up a $\mathbb{C}$-bilinear map

$$H^q(X;\Omega_X^p) \times H^{n-q}(X;\Omega_X^{n-p}) \xrightarrow{\wedge} H^n(X;\Omega_X^n) \xrightarrow[\simeq]{\int_X} \mathbb{C}$$

$$\underset{(p,q)}{([\alpha]}, \underset{(n-p,n-q)}{[\beta])} \longmapsto \underset{(n,n)}{[\alpha\wedge\beta]} \mapsto \int_X \alpha\wedge\beta$$

Together with the existence of unique harmonic representatives (see §2), this pairing can be used to prove the following result:

Theorem (Serre duality)

If X is a compact connected complex analytic manifold, then $\dim_{\mathbb{C}} H^p(X;\Omega_X^p) < +\infty$ and

$$H^q(X;\Omega_X^p) \simeq H^{n-q}(X;\Omega_X^{n-p})^*$$

via the pairing above.

Dolbeault's theorem also tells us that, if $p+q > 2n$, then $H^q(X; \Omega_X^p) = 0$.

Notation $\underbrace{H^{p,q}(X)}_{\text{Dolbeault cohomology}} := H^q(X; \Omega_X^p)$

The $h^{p,q}(X) := \dim H^{p,q}(X)$ are called the Hodge numbers of X. By Serre duality, they satisfy the relation

$$h^{p,q} = h^{n-p,n-q}.$$

In particular, $h^{n,n} = h^{0,0} = 1$, since $H^0(X; \Omega_X^0) = \mathcal{O}_X(X) = \mathbb{C}$ if X is compact and connected.

$$\begin{array}{ccc} & h^{n,n} & \\ h^{n,0} & \begin{array}{c}\text{Hodge}\\ \text{diamond}\end{array} & h^{0,n} \\ & h^{0,0} & \end{array}$$

Example If $\dim_{\mathbb{C}} X = 1$ (compact Riemann surface), then

$$h^{1,0} = \dim H^1(X; \mathcal{O}_X) = \dim H^0(X; \Omega_X^1) = h^{0,1}$$

is the genus of X.

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

2. Harmonic representatives

Recall the statement of the Hodge decomposition theorem: (X compact Kähler)

$$\forall k \geq 0, \quad H^k(X;\mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X;\Omega^p_X)$$

locally constant sheaf

locally free $\mathcal{O}_X$-module

Point: both sides are computable using $\mathcal{C}^\infty$ differential forms (De Rham & Dolbeault).

Let us now put a Riemannian metric g on (the real $\mathcal{C}^\infty$ manifold) X.

The De Rham operator

$$d : \mathcal{E}^k(X) \longrightarrow \mathcal{E}^{k+1}(X)$$

then has an adjoint operator

$$d^* : \mathcal{E}^k(X) \longrightarrow \mathcal{E}^{k-1}(X)$$

which is defined as follows:

1. First, put on $\underbrace{\Lambda^k T^*X}_{\mathcal{C}^\infty\text{-cotangent space}}$ the Riemannian metric $(\cdot|\cdot)$ induced by that of X (locally, the k-forms $du_{i_1} \wedge \dots \wedge du_{i_k}$ form an orthonormal basis).

2. Then use that to define a $\mathcal{C}^\infty(X)$-bilinear form on the space of sections of $\Lambda^k T^* X$, which is $\mathcal{E}^k(X)$:

$$\begin{aligned} \mathcal{E}^k(X) \times \mathcal{E}^k(X) &\longrightarrow \mathcal{C}^\infty(X) \\ (\alpha, \beta) &\longmapsto (\alpha|\beta) : \begin{cases} X \to \mathbb{R} \\ x \mapsto (\alpha_x|\beta_x) \end{cases} \end{aligned}$$

3. Next, using that X is compact and oriented, define an $\mathbb{R}$-bilinear form on $\mathcal{E}^k(X)$:

this is a pos. def. symmetric bilinear form, called the L²-metric on $\mathcal{E}^k(X)$.

$$\left\{ \begin{aligned} \mathcal{E}^k(X) \times \mathcal{E}^k(X) &\longrightarrow \mathbb{R} \\ (\alpha, \beta) &\longmapsto (\alpha|\beta)_{L^2} = \int_X (\alpha|\beta) \, \mathrm{vol}_X \end{aligned} \right.$$

4. Finally, define the L^2-adjoint of d by the relation:

$\forall \alpha \in \mathcal{E}^k(X), \forall \beta \in \mathcal{E}^{k+1}(X),$

$$(d\alpha \mid \beta)_{L^2} = (\alpha \mid d^*\beta)_{L^2}.$$

⚠ $$(f\alpha \mid \beta)_{L^2} \neq f(\alpha \mid \beta)_{L^2}.$$

<u>Exercise</u> Find a way to compute $d^*(f\beta)$.

(Indication: use $*: \mathcal{E}^k(X) \xrightarrow{\simeq} \mathcal{E}^{n-k}(X)$ [as defined in Voisin 5.1.1, p. 117] to show first that

$$(\alpha \mid \beta)_{L^2} = \int_X \alpha \wedge *\beta$$

and

$$d^*\beta = (-1)^{|\beta|} *^{-1} d * \quad).$$

Definition The Laplace operator associated to d by the choice of g is

$$\Delta_d := dd^* + d^*d .$$

Proposition $\forall \alpha, \beta \in \Omega^k(X)$, one has

$$(\alpha \mid \Delta_d \beta)_{L^2} = (d\alpha \mid d\beta)_{L^2} + (d^*\alpha \mid d^*\beta)_{L^2} = (\Delta_d \alpha \mid \beta)_{L^2}.$$

In particular, $\Delta_d^* = \Delta_d$ and $\Delta_d \beta = 0$ iff $d\beta = d^*\beta = 0$.

Definition A k-form α such that $\Delta_d \alpha = 0$ is called d-harmonic.

Theorem (Hodge)

For all $k \geq 0$, the map

$$\begin{array}{ccc} \mathrm{Harm}^k_d(X) := \operatorname{Ker} \Delta_d|_{\mathcal{E}^k(X)} & \longrightarrow & H^k_{DR}(X) \\ \alpha & \longmapsto & [\alpha] \end{array}$$

d-closed by the proposition $\dashleftarrow \alpha$

is an isomorphism of $\mathbb{R}$-vector spaces.

i.e. each De Rham cohomology class has a <u>unique harmonic representative</u>.

Similarly, one can define L^2-adjoints ∂^* and $\bar\partial^*$. For instance,

$$\bar\partial^* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X)$$

There is an associated Laplacian

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

and each Dolbeault cohomology class has a unique harmonic representative:

$$\begin{aligned} \mathrm{Harm}^{p,q}_{\bar\partial}(X) &\xrightarrow{\simeq} H^{p,q}(X) \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

Observation

$$\begin{array}{ccc} \mathcal{E}^k(X) \otimes_{\mathbb{R}} \mathbb{C} & = \bigoplus_{p+q=k} & \mathcal{A}^{p,q}(X) \\ \cup & & \cup \\ \mathrm{Harm}^k_d(X) \otimes_{\mathbb{R}} \mathbb{C} & & \mathrm{Harm}^{p,q}_{\bar\partial}(X) \end{array}$$

3. Kähler identities

In §2, the definition of Δ_d and $\Delta_{\bar\partial}$ depend on the choice of a metric, chosen arbitrarily. The key to the Hodge decomposition theorem is that, for a certain type of metric, the two Laplace operators are related.

Definition Let $(X, \mathcal{O}_X)$ be a complex analytic manifold. A Riemannian metric g on (the underlying real manifold of) X is called a Kähler metric if $g_x(iv, iw) = g_x(v,w)$ and the non-degenerate 2-form defined, for all $v, w \in T_x X$, by $\omega_x(v,w) := g_x(iv, w)$ is closed, i.e. if $d\omega = 0$. ω is called the Kähler form

Not all complex analytic manifolds admit Kähler metrics, but $\mathbb{CP}_n$ does. As a consequence, all complex projective manifolds admit a Kähler metric.

Theorem Let $(X, \mathcal{O}_X)$ be a compact Kähler manifold. Then

$$\Delta_d = 2\,\Delta_{\bar\partial}.$$

In particular, $\ker \Delta_d = \ker \Delta_{\bar\partial}$.

⇝ the proof is based on the Kähler identities [Voisin p. 134 sqq].

Corollary $\mathrm{Harm}^k_d(X) = \bigoplus_{p+q=k} \mathrm{Harm}^{p,q}_{\bar\partial}(X)$

in $\mathcal{E}^k(X) \otimes_{\mathbb{R}} \mathbb{C}$.

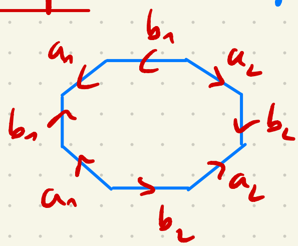
Therefore, we have:

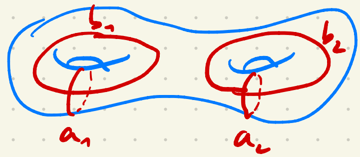
(via Hodge) $$H^k_{DR}(X) \otimes_{\mathbb{R}} \mathbb{C} \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

$$H^k(X;\mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X;\Omega^p_X)$$

(via De Rham and Dolbeault)

Example (compact Riemann surfaces)





$$H^0(X;\mathbb{C}) = \mathbb{C}$$
$$H^1(X;\mathbb{C}) = \mathbb{C}^{2g}$$
$$H^2(X;\mathbb{C}) = \mathbb{C}$$

$$H^0(X;\mathcal{O}_X) = \mathbb{C} = H^1(X;\Omega^1_X)$$
$$H^1(X;\mathcal{O}_X) = \mathbb{C}^g = H^0(X;\Omega^1_X)$$

Poincaré polynomial

$$P_t(X) = 1 + 2gt + t^2$$
$$b_0 = 1$$
$$b_1 = 2g$$
$$b_2 = 1$$

Hodge polynomial

$$H_{u,v}(X) = 1 + gu + gv + uv$$
$$h^{0,0} = h^{1,1} = 1$$
$$h^{1,0} = h^{0,1} = g$$

$$b_k = \sum_{p+q=k} h^{p,q}$$

$$P_t(X) = H_{t,t}(X)$$